Q1.

**Example 6 :** Evaluate the determinant **Determinants**

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$ where ω is a cube root of unity.

**Solution :** $\begin{vmatrix} 1 & \omega & 2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

$1 + \omega + \omega^2$

$$= \begin{vmatrix} 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & 1 \\ 1+\omega+\omega^2 & 1 & \omega \end{vmatrix} \quad (\text{By } C_1 \to C_1 + C_2 + C_3)$$

$$= \begin{vmatrix} 0 & \omega & \omega^2 \\ 0 & \omega^2 & 1 \\ 0 & 1 & \omega \end{vmatrix} \quad (\because 1 + \omega + \omega^2 = 0)$$

$= 0$ [$\because C_1$ consists of all zero entries].

2. Using determinant, find the area of the triangle whose vertices are (− 3, 5), (3, − 6) and (7, 2).

$$\Delta = \frac{1}{2} 1 \begin{vmatrix} -3 & 5 & 1 \\ 3 & -6 & 1 \\ 10 & 2 & 1 \end{vmatrix}$$

$$= \frac{1}{2} 1 \begin{vmatrix} -3 & 5 & 1 \\ 6 & -11 & 0 \\ 10 & -3 & 0 \end{vmatrix} \quad (\text{By applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1)$$

$$= \frac{1}{2} 1 \, |-18 + 110)|$$

$$= \frac{1}{2} \times 92 = 46 \text{ square units}$$

3. Use the principle of mathematical induction to show that $2 + 2^2 + \ldots + 2^n = 2^{n+1} - 2$ for every natural number n.

**Solution :** Let $P_n$ denote the statement

$2 + 2^2 + \ldots\ldots\ldots\ldots + 2^n = 2^{n+1} - 2$

When $n = 1$, $P_n$ becomes

$2 = 2^{1+1} - 2$ or $2 = 4 - 2$

This shows that the result holds for $n = 1$.

Assume that $P_k$ is true for some $k \in \mathbf{N}$.

That is, assume that

$2 + 2^2 + \ldots\ldots\ldots + 2^k = 2^{k+1} - 2$

We shall now show that truth of $P_k$ implies the truth of $P_{k+1}$ is

$$2 + 2^2 + \ldots\ldots\ldots + 2^k + 2^{k+1} = 2^{k+1} - 2 \qquad (1)$$

LHS of (1) $= 2 + 2^2 + \ldots\ldots\ldots + 2^k + 2^{k+1}$

$= (2^{k+1} - 2) + 2^{k+1}$ [induction assumption]

$= 2^{k+1}(1 + 1) - 2$

$= 2^{k+1}\, 2 - 2 = 2^{k+2} - 2$

$=$ RHS of (1)

This shows that the result holds for $n = k+1$; therefore, the truth of $P_k$ implies the truth of $P_{k+1}$. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number $n$.

**4. Find the sum of all integers between 100 and 1000 which are divisible by 9.**

**Solution :** The first integer greater than 100 and divisible by 9 is 108 and the integer just smaller than 1000 and divisible by 9 is 999. Thus, we have to find the sum of the series.

$108 + 117 + 126 + \ldots\ldots\ldots\ldots + 999.$

Here $t_1 = a = 108$, $d = 9$ and $l = 999$

Let $n$ be the total number of terms in the series be $n$. Then

$999 = 108 + 9(n-1) \Rightarrow 111 = 12 + (n-1) \Rightarrow n = 100$

Hence, the required sum $= \frac{n}{2}(a + l) = \frac{100}{2}(108 + 999)$

$= 50\,(1107) = 55350.$

5. Check the continuity of the function f(x) at x = 0 :

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

⑤ $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$

Since $|x| = \begin{cases} x & x > 0 \\ -x, & x < 0 \end{cases}$

$\therefore f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

So, $\lim_{x \to 0+} f(x) = \lim_{x \to 0} (1) = 1$ and

$\lim_{x \to 0-} f(x) = \lim_{x \to 0} (-1) = -1$

Hence $f$ is not continous at $x = 0$

6. If $y = \frac{\ln x}{x}$, show that $\frac{d^2y}{dx^2} = \frac{2 \ln x - 3}{x^3}$

(6)

$$y = \frac{\ln x}{x} \quad \text{or} \quad \frac{dy}{dx} = \ln x \cdot \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\cdot\frac{d}{dx}(\ln x)$$

$$\text{or} \quad \frac{dy}{dx} = \ln x \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{x}\cdot\frac{1}{x}$$

$$= -\frac{1}{x^2}\cdot \ln x + \frac{1}{x^2}$$

$$= \frac{1}{x^2}\cdot(1 - \ln x)$$

Then 2nd order derivative

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d}{dx}(1 - \ln x) + (1 - \ln x)\cdot\left(-\frac{2}{x^3}\right)$$

$$= \frac{1}{x^2}\cdot\left(-\frac{1}{x}\right) - \frac{2}{x^3}(1 - \ln x)$$

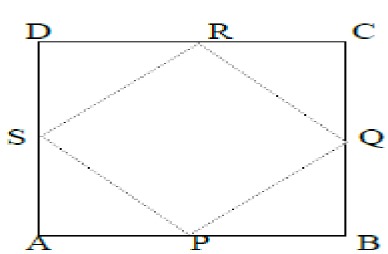
$$= -\frac{1}{x^3} - \frac{2}{x^3} + \frac{2\ln x}{x^3}$$

$$= \frac{-1 - 2 + 2\ln x}{x^3}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-3 + 2\ln x}{x^3} = \frac{2\ln x - 3}{x^3} \quad \text{(Proved)}$$

**7. If the mid-points of the consecutive sides of a quadrilateral are joined, then show (by using vectors) that they form a parallelogram.**

**Solution :** Let $\vec{a}$ , $\vec{b}$ , $\vec{c}$ $\vec{d}$ be the positon vectors of the vertices A, B, C, D of the quadrilateral ABCD. Let *P, Q, R, S* be the mid-points of sides AB, BC, CD, DA respectively. Then the position vectors of P, Q, R and S are $\frac{1}{2}(\vec{a} + \vec{b}), \frac{1}{2}(\vec{b} + \vec{c}), \frac{1}{2}(\vec{c} + \vec{d})$ and $\frac{1}{2}(\vec{d} + \vec{a})$ respectively.




$$Now, \overrightarrow{PQ} = \overrightarrow{PQ} - \overrightarrow{PQ} = \frac{1}{2}(\vec{b}+\vec{c}) - \frac{1}{2}(\vec{a}+\vec{b}) = \frac{1}{2}(\vec{c}-\vec{a})$$

or $\quad \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} \quad (\because \overrightarrow{CA} = -\overrightarrow{AC})$

$\therefore \quad \overrightarrow{PQ} = \overrightarrow{SR}$

$\Rightarrow \quad PQ = SR$ and also PQ || SR.

Since a pair of opposite sides are equal and parallel, therefore, PQRS is a parallelogram.

8. Find the scalar component of projection of the vector

$\vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k}$ on the vector $\vec{b} = 2\hat{i} - 2\hat{j} - \hat{k}$.

**Solution :** Scalar projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

Here, $\vec{a}.\vec{b} = 2.2 + 3(-2) + 5(-1) = -7$

and $|\vec{b}| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$

$\therefore$ Scalar projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{-7}{3}$

**9. Solve the following system of linear equations using Cramer's rule:** x + y = 0, y + z = 1, z + x = 3

(b) Here,

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} \quad \text{[Applying } C_2 \rightarrow C_2 - C_1\text{]}$$

$= 2$ (Expanding along $R_1$)

Since $\Delta \neq 0$, $\therefore$ the given system has unique solution, Now, $\Delta x = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = 2$

$\Delta y = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = -2$

and $\Delta z = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 4$

Hence by Cramer's Rule

$x = \frac{\Delta x}{\Delta} = \frac{2}{2} = 1$

$y = \frac{\Delta y}{\Delta} = \frac{-2}{2} = -1$ and

$z = \frac{\Delta z}{\Delta} = \frac{4}{2} = 2$

10. If A=$\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$, B=$\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, Find a and b.

(4 Marks)

We have $(A + B)^2 = (A + B)(A + B)$

= (A+ B ) A + ( A + B ) B (Distributive Law)

= A A + BA + AB + BB

$= A^2 + BA + AB + B^2$

Therefore, $(A + B)^2 = A^2 + B^2$

$\Rightarrow A^2 + BA + AB + B^2 = A^2 + B^2$

$\Rightarrow BA + AB = 0.$

Thus, we must find $a$ and $b$ such that BA + AB = 0.

We have BA $= \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix}$

and AB $= \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}$

Therefore,

BA + AB $= \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} + \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}$

$= \begin{bmatrix} 2a-b+2 & -a+1 \\ 2a-2 & -b+4 \end{bmatrix}$

But BA + AB = 0

$\Rightarrow 2a - b + 2 = 0, -a + 1 = 0, 2a - 2 = 0, -b + 4 = 0$

$\Rightarrow a = 1, b = 4$

11. Reduce the matrix A(given below) to normal form and hence find its rank.
(4 Marks)

$$A = \begin{bmatrix} 5 & 3 & 8 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

**Solution :** $A = \begin{bmatrix} 5 & 3 & 8 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_3$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 5 & 3 & 8 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - 5R_1$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 8 & 8 \end{bmatrix}$$

Applying elementary row operations $R_1 \rightarrow R_1 + R_2$ and $R_3 \rightarrow R_3 - 8R_2$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we apply elementary column operation $C_3 \rightarrow C_3 - C_2$, to get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Again, applying $C_3 \rightarrow C_3 - C_1$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have thus reduced A to normal form.

Also, note that the rank of a matrix remains unaltered under elementary operations.

Thus, rank of A in above example is 2 because rank of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 2.

In this regard, we state following theorem without proof.

**Theorum :** Every matrix of rank r is equivalent to the matrix $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

12. Show that n(n+1) (2n+1) is a multiple of 6 for every natural number n.

**Solution :** Let $P_n$ denote the statement $n(n + 1)(2n + 1)$ is a multiple of 6.
When $n = 1$, $P_n$ becomes $1(1 + 1)((2)(1) + 1) = (1)(2)(3) = 6$ is a multiple of 6.
This shows that the result is true for $n = 1$.

Assume that $P_k$ is true for some $k \in \mathbf{N}$. That is assume that $k(k + 1)(2k + 1)$ is a mutliple of 6.
Let $k(k + 1)(2k + 1) = 6m$ for some $m \in \mathbf{N}$.
We now show that the truth of $P_k$ implies the truth of $P_{k+1}$, where $P_{k+1}$ is
$(k + 1)(k + 2)[2(k + 1) + 1] = (k + 1)(k + 2)(2k + 3)$ is a multiple of 6.

We have
$(k + 1)(k + 2)(2k + 3)$

$$= (k + 1)(k + 2)[(2k + 1) + 2]$$
$$= (k + 1)[k(2k + 1) + 2(2k+1) + 4)]$$
$$= (k + 1)[k(2k + 1) + 6(k + 1)]$$
$$= k(k + 1)(2k + 1) + 6(k + 1)^2$$
$$= 6m + 6(k + 1)^2 = 6[m + (k + 1)^2]$$

Thus $(k + 1)(k + 2)(2k + 3)$ is multiple of 6.

This shows that the result holds for $n = k+1$; therefore, the truth of $P_k$ implies the truth of $P_{k+1}$. The two steps required for a proof by mathematical induction have been completed, so our statement is true for each natural number $n$.

13. Find the sum of an infinite G.P. whose first term is 28 and fourth term is $\frac{4}{49}$.
(4 Marks)

$$a = 28, \qquad ar^3 = \frac{4}{49}$$

$$\Rightarrow r^3 = \frac{4}{49} \times \frac{1}{28} = \frac{1}{7^3}$$

$$\Rightarrow r = 1/7$$

$$\text{Thus, S} = \frac{a}{1 - r} = \frac{28}{1 - 1/7} = \frac{28 \times 7}{6} = \frac{98}{3}$$

14. Use De Moivre's theorem to find $(\sqrt{3} + i)^3$.

**Solution** : We first put $\sqrt{3} + i$ in the polar form.

Let $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad \sqrt{3} = r\cos\theta$ and $1 = r\sin\theta$

$\Rightarrow \quad (\sqrt{3})^2 + 1^2 = r^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow \quad r^2 = 4 \quad \Rightarrow \quad r = 2$

Thus, $\sqrt{3} + i = 2(\cos\theta + i\sin\theta)$

$\Rightarrow \quad \sqrt{3} = 2\cos\theta$ and $1 = 2\sin\theta$

$\Rightarrow \quad 2\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = 30°$.

Now, $(\sqrt{3} + i)^3 = [2\cos(30°) + i\sin(30°)]^3$

$= 2^3 [\cos(30°) + I\sin(30°)]^3$

$= 8 [\cos(3 \times 30°) + i\sin(3 \times 30°))]$ [De Moivre's theorem]

$= 8(\cos 90° + i\sin 90°) = 8(0 + i)$

$= 8i$

15. If 1, $\omega$, $\omega^2$ are cube roots unity, show that $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = 49$.

(ii) Since $\omega^{10} = (\omega^3)^3\ \omega = \omega$

and $\omega^{11} = (\omega^3)^3\ \omega^2 = \omega^2$,

Thus $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$

$= [(2-\omega)(2-\omega^2)]^2$

$= [4 - 2\omega - 2\omega^2 + \omega^3]^2$

$= [4 - 2(\omega + \omega^2) + 1]^2$

$= [4 - 2(-1) + 1]^2$ $\qquad [\because \omega + \omega^2 = -1]$

$= 7^2 = 49$

**16. Solve the equation 2x3 - 15x2 + 37x – 30 = 0, given that the roots of the equation are in A.P.**

**Example 6** : Solve the equation

$$2x^3 - 15x^4 + 37x - 30 = 0 \qquad (1)$$

If the roots of the equation are in A.P.

**Solution** : Recall three numbers in A.P. can be taken as $\alpha - \beta, \alpha, \alpha + \beta$.

If $\alpha - \beta, \alpha, \alpha + \beta$ are roots of (1), then $(\alpha - \beta) + \alpha + (\alpha + \beta) = 15/2 \Rightarrow 3\alpha = 15/2$

$\Rightarrow \alpha = 5/2$

Next,

$\alpha(\alpha - \beta) + \alpha(\alpha + \beta)\ (\alpha - \beta)(\alpha + \beta) = 37/2$

$\Rightarrow \alpha^2 - \alpha\beta + \alpha^2 + \alpha\beta + \alpha^2 - \beta^2 = 37/2$

$\Rightarrow 3\alpha^2 - \beta^2 = 37/2$

$\Rightarrow \beta^2 = 3\alpha^2 - \frac{37}{2} = 3 \times \frac{25}{4} - \frac{37}{2} = \frac{1}{4}$

$\Rightarrow \beta = \pm\frac{1}{2}$

When $\beta = ½$, the roots are

$\frac{5}{2} - \frac{1}{2}, \frac{5}{2}, \frac{5}{2} + \frac{1}{2}$, or $2, \frac{5}{2}, 3$

When $\beta = -\frac{1}{2}$, the roots are 3, 5/2 2.

It is easily to check that these are roots of (1).

**17. A young child is flying a kite which is at height of 50 m. The wind is carrying the kite horizontally away from the child at a speed of 6.5 m/s. How fast must the kite string be let out when the string is 130m ?**

**Solution :** Let $h$ be the horizontal distance of the kite from the point directly over the child's head 5. Let $l$ be the length of kite string from the child to the kite at time $t$. [See Fig. 1] Then

$$l^2 = h^2 + 50^2$$

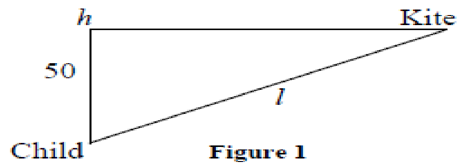



**Figure 1**

Differentiating both the sides with respect to $t$, we get

$$2l\,\frac{dl}{dt} = 2h\,\frac{dh}{dt} \text{ or } l\,\frac{dl}{dt} = h\,\frac{dh}{dt}.$$

We are given $\frac{dh}{dt} = 6.5\ m/s$. We are interested to find $dl/dt$ when $l = 130$.
But when $l = 130$, $h^2 = l^2 - 50^2 = 130^2 - 50^2 = 14400$ or $h = 120$.

$$\text{Thus, } \frac{dl}{dt} = \frac{120}{130} \times 6.5 = 5\ m/s.$$

This shows that the string should be let out at a rate of 6 m/s.

18. Using first derivative test, find the local maxima and minima of the function $f(x) = x^3 - 12x$. (4 Marks)

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(i) $f(x) = x^3 - 12x$

Differentiating w.r.t. $x$, we get

$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 2(x-2)(x+2)$

Setting $f'(x) = 0$, we obtain $x = 2, -2$ Thus, $x = -2$, and $x = 2$ are the only critical numbers of $f$. Fig. 35 shows the sign of derivative $f'$ in three intervals.

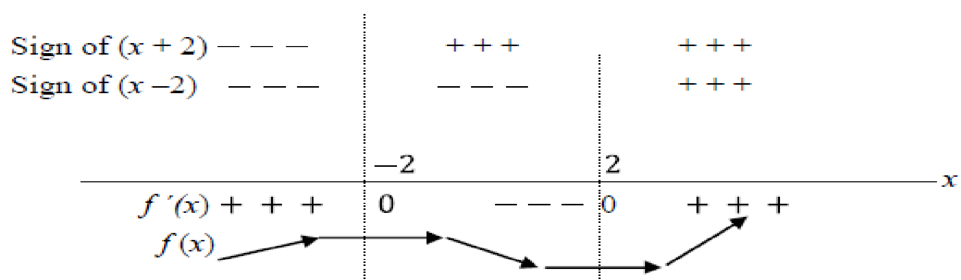



**Figure 35**

From figure 35 it is clear that if $x < -2$, $f'(x) > 0$; if $-2 < x < 2$, $f'(x) < 0$ and if $x > 2$, $f'(x) > 0$.

Using the first derivative test, we conclude that

$f(x)$ has a local maximum at $x = -2$ and a local minimum at $x = 2$.

Now, $f(-2) = (-2)^3 - 12(-2) = -8 + 24 = 16$ is the value of local maximum at $x = -2$ and $f(2) = 2^3 - 12(2) = 8 - 24 = -16$ is the value of the local minimum at $x = 2$.

19. Evaluate the integral $I = \int \frac{x^2}{(x+1)^3} dx$

**Solution :** To evaluate an integral of the form

$$\int \frac{P(x)}{(a+bx)^r} dx, \text{ we put } a + bx = t.$$

So, we put $x + 1 = t \Rightarrow dx = dt$

$$\text{and I} = \int \frac{(t+1)^2}{t^3} dt = \int \frac{t^2 + 2t + 1}{t^3} dt$$

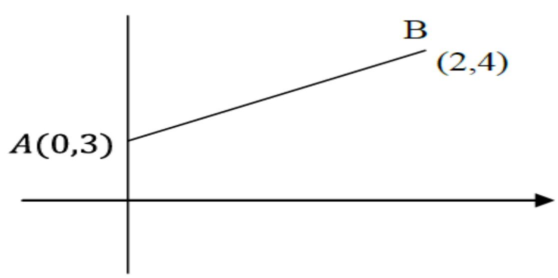
$$= \int \left(\frac{1}{t} - 2t^{-2} + t^{-3}\right) dt$$

$$= \log|t| - \frac{2t^{-1}}{-1} + \frac{t^{-2}}{-2} + c$$

$$= \log|t| + \frac{2}{t} - \frac{1}{2t^2} + c$$

$$= \log|x+1| - \frac{2}{x+1} + \frac{1}{2(x+1)^2} + c$$

20. Find the length of the curve $y = 3 + \frac{x}{2}$ from (0, 3) to (2, 4).




We have

$$\frac{dy}{dx} = \frac{1}{2}$$

Required length

$$= \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^2 \sqrt{1 + \frac{1}{4}}\, dx = \frac{\sqrt{5}}{2} \int_0^2 dx = \frac{\sqrt{5}}{2} \left[x\right]_0^2 = \sqrt{5} \text{ units}$$